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THE DEFORMATION OF A CONE OF TRANSVERSALLY ISOTROPIC MATERIAL BY THE ACTION OF TEMPERATURE AND AN AXIAL FORCE†

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A self-similar solution of the problem of the uniform heating of an elastic cone of transversally isotropic material with curvilinear cylindrical anisotropy is obtained in a cylindrical system of coordinates. It is assumed that the cone has a small vertex angle. A solution of the problem of the deformation of the cone in question due to the action of an axial force is given. A feature of the problems is the fact that the axes of physical and geometrical symmetry do not coincide.

A solution of the problem of the deformation of a cone of transversally isotropic material due to the action of an axial force was derived in [1, 2]. The case when the axes of physical and geometrical symmetry of the body are collinear was considered in [1], and the case of a material with spherical anisotropy, when the axis of symmetry of the material is directed along a spherical radius, was considered in [2].

1. A CONE OF TRANSVERSALLY ISOTROPIC MATERIAL ACTED UPON BY A UNIFORMLY DISTRIBUTED TEMPERATURE

We will consider the self-similar solution of the equations of the theory of elasticity in a cylindrical system of coordinates r, θ, x as it applies to the problem of the uniform heating of an elastic cone of transversally isotropic material with curvilinear cylindrical anisotropy. A feature of the problem is the fact that the plane of isotropy of the material is not perpendicular to the axis of the cone. We will assume that the cone has a small angle at the vertex, i.e. $\operatorname{tg}\varphi \ll 1$, where 2φ is the vertex angle of the cone.

The origin of coordinates is placed at the vertex of the cone, and the x axis is directed along the axis of the cone. The ends of the cone are described by the equations $x = x_0 \geqslant 0$, $x = X > x_0$.

Henceforth we will give the directions r, φ and x the subscripts 1, 2 and 3.

The stresses σ_1, σ_2, σ_3, σ_{13} and the strains ε_1, ε_2, ε_3, $\frac{1}{2}\varepsilon_{13}$ in the material considered are related by the following equations [1]

$$\begin{aligned}
&E_2\varepsilon_1 = k\sigma_1 - kv'(\sigma_2+\sigma_3) + E_2\alpha_1\Delta T\\
&E_2\varepsilon_2 = \sigma_2 - v\sigma_3 - kv'\sigma_1 + E_2\alpha_2\Delta T\\
&E_2\varepsilon_3 = \sigma_3 - v\sigma_2 - kv'\sigma_1 + E_2\alpha_2\Delta T\\
&E_2\varepsilon_{13} = \gamma\sigma_{13}, \quad k = E_2/E_1, \; \gamma = E_2/G
\end{aligned} \tag{1.1}$$

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The axis of symmetry of the transversally isotropic material is directed along axis 1, E_1 and E_2 are the moduli of elasticity in directions 1 and 2, respectively, G is the shear modulus, α_1 and α_2 are the coefficients of thermal expansion along axes 1 and 2, and $\Delta T = \text{const}$ is the temperature of the cone.

The case then $k > 1$ is of interest.

We will seek a solution of the problem in the form

$$\sigma_i = \sigma_i(y), \quad \varepsilon_i = \varepsilon_i(y), \quad i = 1,2,3$$
$$\sigma_{13} = \sigma_{13}(y), \quad \varepsilon_{13} = \varepsilon_{13}(y), \quad y = x/r \tag{1.2}$$

The equations of equilibrium and compatibility of the strains for the case of axisymmetrical deformation in a cylindrical system of coordinates when conditions (1.2) are satisfied can be written in the form

$$-\sigma_1'y + \sigma_{13}' + \sigma_1 - \sigma_2 = 0, \quad -\sigma_{13}'y + \sigma_{13} + \sigma_3' = 0 \tag{1.3}$$

$$-\varepsilon_2'y + \varepsilon_2 - \varepsilon_1 = 0, \quad \varepsilon_2'' = \varepsilon_{13}' + \varepsilon_3'y \tag{1.4}$$

where the prime denotes differentiation with respect to y.

There are no surface loads on the side surface of the cone $y = y_1 = \operatorname{ctg}\varphi$. In this case the boundary conditions on the side surface can be written as follows [1]:

$$\sigma_1(y_1)y_1 = \sigma_{13}(y_1), \quad \sigma_{13}(y_1)y_1 = \sigma_3(y_1) \tag{1.5}$$

The condition for the resultant force acting on the ends $x = x_0$, $x = X$ to be zero has the form

$$\int_0^{x\operatorname{tg}\varphi} \sigma_3 r\,dr = 0 \tag{1.6}$$

We will seek a solution that is bounded as $y \to \infty$ $(r \to 0)$. Then, when the second equations of (1.3) and (1.5) are satisfied, condition (1.6) is satisfied automatically.

We will convert the second equation of the compatibility of the strains into the relation

$$\varepsilon_3'y = -\varepsilon_1'y^{-1} - \varepsilon_{13}' \tag{1.7}$$

Taking into account the fact that the angle φ is small, the solution of Eqs (1.1), (1.3) and (1.7) and the first equation of (1.4) can be written in the form of series [3]

$$\sigma_i = D_1 y^{-\omega+1} \sum a_{i,m} y^{-2m} + B_i / [(1-\nu^2)(\omega^2 - 1)], \quad i = 1,2,3$$
$$\sigma_{13} = D_1 y^{-\omega} \sum a_{4,m} y^{-2m}$$
$$B_1 = B_2 = -[P + (\nu - k\nu')D_2], \quad B_3 = D_2(k-1) - (\nu + k\nu')P$$
$$P = E_2(\alpha_1 - \alpha_2)\Delta T, \quad \omega = [(k - (k\nu')^2)/(1-\nu^2)]^{1/2}$$
$$a_{1,0} = 1, \quad a_{2,0} = \omega, \quad a_{3,0} = \nu\omega + k\nu', \quad a_{4,0} = a_{3,0}(\omega - 1)/(\omega + 1)$$

where D_1 and D_2 are undetermined constants, and the summation here and henceforth is carried out with respect to m from 0 to ∞.

Note that for a material with a pronounced anisotropy we can assume that $\omega > 1$ when $k > 1$.

A method of finding ω and $a_{i,0}$ $(i = 1, \ldots, 4)$ will be described below.

The coefficients $a_{j,m}$ $(j = 1, \ldots, 4;\ m \geqslant 1)$ are defined by the following recurrent relations

$$(\omega + 2m)a_{1,m} - a_{2,m} = (\omega + 2m - 2)a_{4,m-1}$$
$$(\omega + 2m)(-k\nu' a_{1,m} + a_{2,m} - \nu a_{3,m}) - k a_{1,m} + k\nu'(a_{2,m} + a_{3,m}) = 0$$

$$(\omega+2m-1)(-k\nu' a_{1,m}-\nu a_{2,m}+a_{3,m})=$$
$$=-(\omega+2m-3)k\{a_{1,m-1}-\nu'(a_{2,m-1}+a_{3,m-1})\}-(\omega+2m-2)\gamma a_{4,m-1}$$
$$(\omega+2m+1)a_{4,m}=(\omega+2m-1)a_{3,m}$$

If we retain only the first terms $a_{i,0}$ in the series $\Sigma a_{i,m} y^{-2m}$, we obtain an asymptotic solution of the problem. It satisfies the second relation of (1.3), the first equation of (1.4) and the equations

$$-\sigma_1' y+\sigma_1-\sigma_2=0, \quad E_2\varepsilon_3=D_2+E_2\alpha_2\Delta T$$

which are asymptotic approximations as $1<\omega<2$ of the first equation of (1.3) and relation (1.7), respectively.

This system of equations can easily be integrated, and as a result one can determine the coefficients $a_{i,0}$ ($i=1, \ldots, 4$) apart from a constant factor D_1, and the quantity ω.

The constants D_1 and D_2 can be found from the boundary conditions (1.5).

A singularity of the solution obtained is described by the factor in front of the series summation sign. In particular, by writing the solution of the problem in the natural variable $\xi=y^{-1}$ we obtain that at the point $\xi=0$ when $1<\omega<2$ the derivatives $d\sigma_i/d\xi$ ($i=1, 2, 3$) tend to infinity.

2. A CONE ACTED UPON BY AN AXIAL FORCE

We will investigate the stress state of an elastic cone ($\mathrm{tg}\varphi \ll 1$) of transversally isotropic material acted upon by oppositely directed axial forces Q applied at the ends $x=x_0\geqslant 0$, $x=X>x_0$.

The relation between the stresses and the strains is given by relations (1.1). As in Section 1 we will consider the case when $k>1$ ($\omega>1$).

We will write the solution of the problem in the form

$$\begin{aligned}&\sigma_i=x^{-2}p_i(y), \quad \varepsilon_i=x^{-2}e_i(y), \quad i=1,2,3\\&\sigma_{13}=x^{-2}p_{13}(y), \quad \varepsilon_{13}=x^{-2}e_{13}(y), \quad y=x/r\end{aligned} \tag{2.1}$$

We will write the equations of equilibrium and compatibility of the strains, taking (2.1) into account, as follows:

$$\begin{aligned}&-p_1' y+p_1-p_2+(p_{13}y^{-2})' y^2=0\\&-p_{13}' y+p_{13}-2p_3 y^{-1}+p_3'=0\\&-e_2' y+e_2-e_1=0\end{aligned} \tag{2.2}$$

$$y^2(e_2 y^{-2})''=(e_{13}y^{-2})' y^2+e_3' y \tag{2.3}$$

The boundary conditions on the side surface of the cone are given by relations (1.5).

Axial forces Q with opposite signs act on the ends of the cone $x=x_0$, $x=X$

$$2\pi\int_0^{\mathrm{ctg}\varphi}\sigma_3 r\,dr=Q$$

Using (2.1) we will write the last condition in the form

$$2\pi\int_\infty^{\mathrm{ctg}\varphi} p_3 y^{-3}\,dy=-Q \tag{2.4}$$

The second equation of equilibrium (2.2) and the second boundary condition (1.5) are satisfied identically if we put

$$\sigma_3 = y\sigma_{13} \tag{2.5}$$

We convert the second equation of compatibility (2.3) into the relation

$$y\{-y^{-3}(e_2+e_1)-e_{13}y^{-2}\}' = e_3' \tag{2.6}$$

Taking into account the small geometrical parameter of the cone $\operatorname{tg}\varphi \ll 1$ we will seek a solution of the first equations of (2.2) and (2.3) and of Eqs (2.5) and (2.6) in the form of series

$$p_i = D_1 y^{-\omega+1}\sum a_{i,m}y^{-2m} + D_2\sum b_{i,m}y^{-2m}, \quad i=1,2,3$$
$$p_{13} = D_1 y^{-\omega}\sum a_{4,m}y^{-2m} + D_2 y^{-1}\sum b_{4,m}y^{-2m}$$

where D_1 and D_2 are constants.

The asymptotic solution

$$p_i = D_1 y^{-\omega+1}a_{i,0} + D_2 b_{i,0}, \quad p_{13} = D_1 y^{-\omega}a_{4,0} + D_2 y^{-1}b_{4,0}$$

is found from the first equation of (2.3), relation (2.5) and the equations

$$-p_1'y + p_1 - p_2 = 0, \quad E_2 e_3 = D_2$$

The last two relations are asymptotic approximations as $y \to \infty$ of the first equation of (2.2) and of Eq. (2.6), respectively.

Integrating this system of equations we obtain

$$a_{1,0}=1, \quad a_{2,0}=\omega, \quad a_{3,0}=\nu\omega+k\nu', \quad a_{4,0}=a_{3,0}$$
$$b_{1,0}=b_{2,0}=-\frac{\nu-k\nu'}{(1-\nu^2)(\omega^2-1)}, \quad b_{3,0}=b_{4,0}=\frac{k-1}{(1-\nu^2)(\omega^2-1)}$$
$$\omega=[(k-(k\nu')^2)/(1-\nu^2)]^{1/2}$$

The coefficients $a_{i,m}(m \geqslant 1)$ are defined in recurrent form in terms of the coefficients $a_{i,m-1}$ $(i=1,\ldots,4)$ Similarly, $b_{i,m}(m \geqslant 1)$ are calculated in terms of the quantities $b_{i,m-1}$ $(i=1,\ldots,4)$.

In order to obtain these recurrent relations we must substitute the solution $p_i(y)$ $(i=1, 2, 3)$, $p_{13}(y)$ calculated above into (2.2), (2.6) and the first equation of (2.3).

3. RESULTS OF CALCULATIONS

Calculations were carried out for $k=3$, $\nu=0.2$, $k\nu'=0.3$, $\gamma=6$, $\operatorname{tg}\varphi=0.25$ and different values of L (the number of the term at which the sums of the series were terminated).

In columns 2–5 of Table 1 we show the values of the dimensionless temperature stresses $\sigma_i^* = \sigma_i/|P|$ $(i=1, 2, 3)$, $\sigma_{13}^*=\sigma_{13}/|P|$ at the points $\xi=0$, $0.5\operatorname{tg}\varphi$, $\operatorname{tg}\varphi(\xi=y^{-1})$ for the case when $\alpha_1>\alpha_2$, $\Delta T=\text{const}<0$.

Note that when $L \geqslant 5$ the results are practically identical.

The stresses decrease monotonically along the ξ coordinate.

Table 1 also shows the values of the self-similar functions $p_i^* = p_i \times 2\pi/Q$ $(i=1, 2, 3)$, $p_{13}^* = p_{13}\times 2\pi/Q$.

The results of calculations of these functions are practically identical for $L \geqslant 11$.

The functions p_2 and p_3 decrease monotonically along the ξ coordinate; the function p_1 attains a minimum value at the point $\xi \approx 0.5$.

Table 1

$\xi/\mathrm{tg}\,\varphi$	σ_1^*	σ_2^*	σ_3^*	$10^2\times\sigma_{13}^*$	p_1^*	p_2^*	p_3^*	p_{13}^*
0	0.5114	0.5114	0.2276	0	2.045	2.045	40.90	0
0.5	0.2032	-0.0243	-0.0340	-0.632	1.154	-0.265	35.43	4.428
1	-0.0044	-0.3821	-0.0700	-1.748	1.599	-0.750	25.58	6.394

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